

GENERALIZED TANAKA-WEBSTER PARALLEL RICCI TENSOR IN COMPLEX TWO-PLANE GRASSMANNIANS

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ABSTRACT. We prove the non-existence of Hopf real hypersurfaces in complex two-plane Grassmannians whose Ricci tensor is parallel with respect to the generalized Tanaka-Webster connection.

1. INTRODUCTION

The generalized Tanaka-Webster connection (from now on, g-Tanaka Webster connection) for contact metric manifolds was introduced by Tanno ([10]) as a generalization of the connection defined by Tanaka in [9] and, independently, by Webster in [11]. The Tanaka-Webster connection is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface M in a Kähler manifold has an (integrable) CR-structure associated with the almost contact structure (ϕ, ξ, η, g) induced on M by the Kähler structure, but, in general, this CR-structure is not guaranteed to be pseudo-Hermitian. Kon considered a g-Tanaka-Webster connection for a real hypersurface of a Kähler manifold (see [4]) by

$$(1.1) \quad \hat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$$

for any X, Y tangent to M , where ∇ denotes the Levi-Civita connection on M , A is the shape operator on M and k is a non-zero real number. In particular, if the real hypersurface satisfies $A\phi + \phi A = 2k\phi$, then the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see [10] and [11]).

Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in \mathbb{C}^{m+2} . It is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure \mathfrak{J} not containing J (see Berndt and Suh [2]). In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ and N a local normal unit vector field on M . Let also A be the shape operator of M associated to N . Then we define the structure vector field of M by $\xi = -JN$. Moreover, if $\{J_1, J_2, J_3\}$ is a local basis of \mathfrak{J} , we define $\xi_i = -J_i N$, $i = 1, 2, 3$. We will call $D^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$.

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M is called Hopf if ξ is principal, that is, $A\xi = \alpha\xi$. Berndt and Suh, [2] proved that if $m \geq 3$, a real hypersurface M of $G_2(\mathbb{C}^{m+2})$ for which both $[\xi]$ and \mathfrak{D}^\perp are A -invariant must be an open part of either (A) a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or (B) a tube around a totally geodesic HP^n in $G_2(\mathbb{C}^{m+2})$. In this second case $m = 2n$.

Let S denote the Ricci tensor of the real hypersurface M . In [6] we proved the non-existence of Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel and commuting Ricci tensor, that is $\nabla S = 0$ and $S\phi = \phi S$ for the structure tensor ϕ .

Moreover, this result was improved by Suh, [7] and [8], who proved that the above non-existence property also can be hold for either parallel or commuting Ricci tensor.

In this paper, related to the parallel Ricci tensor, we will study the corresponding condition using g -Tanaka-Webster connection. That is, we will consider real hypersurfaces for which $\hat{\nabla}_X^{(k)} S = 0$ for any X tangent to M . We obtain the following

Theorem 1.1. *There do not exist connected orientable Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose Ricci tensor is parallel with respect to the g -Tanaka-Webster connection.*

2. PRELIMINARIES

For the study of the Riemannian geometry of $G_2(\mathbb{C}^{m+2})$ see [1]. All the notations we will use from now on are the ones in [2] and [3]. We will suppose that the metric g of $G_2(\mathbb{C}^{m+2})$ is normalized for the maximal sectional curvature of the manifold to be eight. Then the Riemannian curvature tensor \bar{R} of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}
 (2.1) \quad \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\
 &+ \sum_{\nu=1}^3 \{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\} \\
 &+ \sum_{\nu=1}^3 \{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\},
 \end{aligned}$$

where J_1, J_2, J_3 is any canonical local basis of \mathfrak{J} .

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g , and ∇ denotes the Riemannian connection of (M, g) . Let N be a local unit normal field of M and A the shape operator of M with respect to N . The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g) . Furthermore, let J_1, J_2, J_3 be a canonical local basis of \mathfrak{J} . Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M .

Since \mathfrak{J} is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, for any canonical local basis J_1, J_2, J_3 of \mathfrak{J} there exist three local 1-forms q_1, q_2, q_3 such that

$$(2.2) \quad \bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$$

for any X tangent to $G_2(\mathbb{C}^{m+2})$, where subindices are taken modulo 3.

From the expression of the curvature tensor of $G_2(\mathbb{C}^{m+2})$ the Gauss equation is given by

$$(2.3) \quad \begin{aligned} R(X, Y)Z = & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + \sum_{\nu=1}^3 \{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y) - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\ & + \sum_{\nu=1}^3 \{g(\phi_\nu \phi Y, Z)\phi_\nu \phi X - g(\phi_\nu \phi X, Z)\phi_\nu \phi Y\} \\ & - \sum_{\nu=1}^3 \{\eta(Y)\eta_\nu(Z)\phi_\nu \phi X - \eta(X)\eta_\nu(Z)\phi_\nu \phi Y\} \\ & - \sum_{\nu=1}^3 \{\eta(X)g(\phi_\nu \phi Y, Z) - \eta(Y)g(\phi_\nu \phi X, Z)\}\xi_\nu \\ & + g(AY, Z)ZX - g(AX, Z)AY \end{aligned}$$

for any X, Y, Z tangent to M .

From (2.3) the Ricci tensor of M is given by

$$(2.4) \quad \begin{aligned} SX = & (4m+7)X - 3\eta(X)\xi - \sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu \\ & + \sum_{\nu=1}^3 \{\eta_\nu(\xi)\phi_\nu \phi X - \eta(\phi_\nu X)\phi_\nu \xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} \\ & + hAX - A^2X \end{aligned}$$

for any X tangent to M , where h denotes $\text{trace}(A)$.

From (2.4) we can compute, see [6],

$$\begin{aligned}
(2.5) \quad (\nabla_X S)Y &= -3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX \\
&- 3 \sum_{\nu=1}^3 \{q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) + g(\phi_\nu AX, Y)\}\xi_\nu \\
&- 3 \sum_{\nu=1}^3 \eta_\nu(Y) \{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX\} \\
&+ \sum_{\nu=1}^3 \{X(\eta_\nu(\xi))\phi_\nu \phi Y + \eta_\nu(\xi) \{-q_{\nu+1}(X)\phi_{\nu+2}\phi Y \\
&+ q_{\nu+2}(X)\phi_{\nu+1}\phi Y + \eta_\nu(\phi Y)AX - g(AX, \phi Y)\xi_\nu\} \\
&+ \eta_\nu(\xi) \{\eta(Y)\phi_\nu AX - g(AX, Y)\phi_\nu \xi\} - g(\phi AX, \phi_\nu Y)\phi_\nu \xi \\
&+ \{q_{\nu+1}(X)\eta(\phi_{\nu+2}Y) - q_{\nu+2}(X)\eta(\phi_{\nu+1}Y) - \eta_\nu(Y)\eta(AX) \\
&+ \eta(\xi_\nu)g(AY, X)\phi_\nu \xi - \eta(\phi_\nu Y)\{q_{\nu+2}(X)\phi_{\nu+1}\xi \\
&- q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu \phi AX - \eta(AX)\xi_\nu + \eta(\xi_\nu)AX\} \\
&- g(\phi AY, X)\eta_\nu(\xi)\xi_\nu - \eta(Y)X(\eta_\nu(\xi))\xi_\nu - \eta(Y)\eta_\nu(\xi)\nabla_X \xi_\nu\} \\
&+ X(h)AY + h(\nabla_X A)Y - (\nabla_X A^2)Y
\end{aligned}$$

for any X, Y tangent to M , where the subindices are taken modulo 3.

A real hypersurface of type (A) has three (if $r = \frac{\pi}{2\sqrt{8}}$) or four (otherwise) distinct principal curvatures $\alpha = \sqrt{8} \cot(\sqrt{8}r)$, $\beta = \sqrt{2} \cot(\sqrt{2}r)$, $\lambda = -\sqrt{2} \tan(\sqrt{2}r)$, $\mu = 0$, for some radius $r \in (0, \frac{\pi}{\sqrt{8}})$, with corresponding multiplicities $m(\alpha) = 1$, $m(\beta) = 2$, $m(\lambda) = m(\mu) = 2m - 2$. The corresponding eigenspaces can be seen in [2].

A real hypersurface of type (B) has five distinct principal curvatures $\alpha = -2 \tan(2r)$, $\beta = 2 \cot(2r)$, $\gamma = 0$, $\lambda = \cot(r)$, $\mu = -\tan(r)$, for some $r \in (0, \frac{\pi}{4})$, with corresponding multiplicities $m(\alpha) = 1$, $m(\beta) = 3 = m(\gamma)$, $m(\lambda) = 4m - 4 = m(\mu)$. For the corresponding eigenspaces see [2].

3. PROOF OF THE THEOREM

If the Ricci tensor of M is g-Tanaka-Webster parallel we get

$$\begin{aligned}
(3.1) \quad 0 &= (\hat{\nabla}_X^{(k)} S)Y = \hat{\nabla}_X^{(k)} SY - S\hat{\nabla}_X^{(k)} Y \\
&= \nabla_X SY + g(\phi AX, SY)\xi - \eta(SY)\phi AX - k\eta(X)\phi SY \\
&\quad - S\nabla_X Y - g(\phi AX, Y)S\xi + \eta(Y)S\phi AX + k\eta(X)S\phi Y
\end{aligned}$$

for any X, Y tangent to M . This yields

$$\begin{aligned}
(3.2) \quad (\nabla_X S)Y &= -g(\phi AX, SY)\xi + \eta(SY)\phi AX + k\eta(X)\phi SY \\
&\quad + g(\phi AX, Y)S\xi - \eta(Y)S\phi AX - k\eta(X)S\phi Y.
\end{aligned}$$

Thus from (2.5) we obtain

$$\begin{aligned}
& -3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX \\
& -3\sum_{\nu=1}^3\{q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) + g(\phi_\nu AX, Y)\}\xi_\nu \\
& -3\sum_{\nu=1}^3\eta_\nu(Y)\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX\} \\
& +\sum_{\nu=1}^3\{X(\eta_\nu(\xi)\phi_\nu\phi Y + \eta_\nu(\xi)\{-q_{\nu+1}(X)\phi_{\nu+2}\phi Y \\
(3.3) \quad & + q_{\nu+2}(X)\phi_{\nu+1}\phi Y + \eta_\nu(\phi Y)AX - g(AX, \phi Y)\xi_\nu\} \\
& + \eta_\nu(\xi)\{\eta(Y)\phi_\nu AX - g(AX, Y)\phi_\nu\xi\} - g(\phi AX, \phi_\nu Y)\phi_\nu\xi \\
& + \{q_{\nu+1}(X)\eta(\phi_{\nu+2}Y) - q_{\nu+2}(X)\eta(\phi_{\nu+1}Y) - \eta_\nu(Y)\eta(AX) \\
& + \eta(\xi_\nu g(AX, Y))\phi_\nu\xi - \eta(\phi_\nu Y)\{q_{\nu+2}(X)\phi_{\nu+1}\xi \\
& - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX - \eta(AX)\xi_\nu + \eta(\xi_\nu)AX\} \\
& - g(\phi AX, Y)\eta_\nu(\xi)\xi_\nu - \eta(Y)X(\eta_\nu(\xi))\xi_\nu - \eta(Y)\eta_\nu(\xi)\nabla_X\xi_\nu\} \\
& + X(h)AY + h(\nabla_X A)Y - (\nabla_X A^2)Y \\
& = -g(\phi AX, SY)\xi + \eta(SY)\phi AX + k\eta(X)\phi SY \\
& + g(\phi AX, Y)S\xi - ta(Y)S\phi AX - k\eta(X)S\phi Y
\end{aligned}$$

for any X, Y tangent to M .

Lemma 3.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, such that its Ricci tensor is g -Tanaka-Webster parallel. Then either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.*

Proof. We can write $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$, where X_0 is a unit vector field in \mathfrak{D} . Suppose that $A\xi = \alpha\xi$ and that $\eta(X_0)\eta(\xi_1) \neq 0$. As $\xi(\eta_1(\xi)) = g(\xi, \nabla_\xi\xi_1)$, if we take $X = \xi$ and $Y = \phi X_0$ in (3.3) we get, bearing in mind that $\eta_\nu(\phi X_0) = 0$, $\nu = 1, 2, 3$.

$$\begin{aligned}
& 3\alpha\eta(\xi_1)\eta(X_0)\xi_1 + \eta(\nabla_\xi\xi_1)\eta(X_0)\phi_1\xi - \eta(\nabla_\xi\xi_1)\phi_1X_0 \\
& + \eta_1(\xi)\{-q_2(\xi)\phi_3\phi^2X_0 + q_3(\xi)\phi_2\phi^2X_0 + \eta(X_0)\eta(\xi_1)\alpha\xi\} \\
& + \{q_2(\xi)\eta(\phi_3\phi X_0) - q_3(\xi)\eta(\phi_2\phi X_0)\}\phi_1\xi \\
(3.4) \quad & + \{q_3(\xi)\eta(\phi_1\phi X_0) - q_1(\xi)\eta(\phi_3\phi X_0)\}\phi_2\xi \\
& + \{q_1(\xi)\eta(\phi_2\phi X_0) - q_2(\xi)\eta(\phi_1\phi X_0)\}\phi_3\xi \\
& - \eta(X_0)\eta(\xi_1)\{q_3(\xi)\phi_2\xi - q_2(\xi)\phi_3\xi - \alpha\xi_1 + \alpha\eta(\xi_1)\xi\} \\
& + \xi(\alpha)A\phi X_0 + h(\nabla_\xi A)\phi X_0 - (\nabla_\xi A^2)\phi X_0 \\
& = k\phi S\phi X_0 + kSX_0 - k\eta(X_0)S\xi.
\end{aligned}$$

Now we have $S\xi = (4m+4+h\alpha-\alpha^2)\xi - 4\eta(\xi_1)\xi_1$, $\eta(\phi_1\phi X_0) = -g(\phi X_0, \phi\xi_1) = \eta(X_0)\eta(\xi_1)$ and $\eta(\phi_\nu\phi X_0) = 0$, $\nu = 2, 3$. Introducing these equalities in (3.4) and taking its scalar product with ξ we obtain

$$(3.5) \quad 4(\alpha - k)\eta^2(\xi_1)\eta(X_0) = g(\phi X_0, (\nabla_\xi A^2)\xi) - hg(\phi X_0, (\nabla_\xi A)\xi).$$

As $g(\phi X_0, (\nabla_\xi A)\xi) = g(\phi X_0, \nabla_\xi \alpha \xi) = \alpha g(\phi X_0, \phi A\xi) = 0$ and the same is true for the other term in the right of the equality in (3.5) we arrive to

$$(3.6) \quad 4(\alpha - k)\eta^2(\xi_1)\eta(X_0) = 0.$$

As we suppose $\eta(\xi_1)\eta(X_0) \neq 0$, we get $\alpha = k$. Thus α is constant. From Berndt and Suh, [2], we know that M being Hopf, for any $Y \in TM$, $Y(\alpha) = \xi(\alpha)\eta(Y) - 4\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(\phi Y)$. This yields $\phi_1\xi = 0$, giving a contradiction.

Therefore $\eta(X_0)\eta(\xi_1) = 0$ and we obtain the result. \square

By Lee and Suh, [5], if $\xi \in \mathfrak{D}$, M is locally a type (B) real hypersurface.

Consider the case $\xi \in \mathfrak{D}^\perp$. Then we have

Lemma 3.2. *Let M be a Hopf real hypersurface of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Suppose that the Ricci tensor of M is g -Tanaka-Webster parallel and $\xi \in \mathfrak{D}^\perp$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. As $\xi \in \mathfrak{D}^\perp$ we can suppose that $\xi = \xi_1$. As M is Hopf $g(A\xi_1, X) = 0$ for any $X \in \mathfrak{D}$. Thus we must prove that $g(A\xi_\nu, X) = 0$, $\nu = 2, 3$.

Take $X \in \mathfrak{D}$, $Y = \xi$ in (3.3). We obtain

$$(3.7) \quad \begin{aligned} & -3\phi AX + 4g(AX, \phi_2\xi_1)\xi_2 + 4g(AX, \phi_3\xi_1)\xi_3 + X(h)A\xi \\ & + h(\nabla_X A)\xi - (\nabla_X A^2)\xi = \eta(S\xi)\phi AX - S\phi AX. \end{aligned}$$

From (2.4) we have $S\xi = (4m + h\alpha - \alpha^2)\xi$ and $S\xi_2 = (4m + 6)\xi_2 + hA\xi_2 - A^2\xi_2$. If we take the scalar product of (3.7) and ξ_2 and use these expressions we obtain $5g(AX, \xi_3) = 6g(AX, \xi_3)$. That is

$$(3.8) \quad g(A\xi_3, X) = 0$$

for any $X \in \mathfrak{D}$. Similarly we obtain

$$(3.9) \quad g(A\xi_2, X) = 0$$

for any $X \in \mathfrak{D}$. From (3.8) and (3.9) the result follows. \square

From Lemma 3.1 and Lemma 3.2 we know that M is locally congruent to a real hypersurface either of type (A) or of type (B).

Suppose that M is of type (A). Remember that $A\xi = \alpha\xi$, $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, with $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\beta = \sqrt{2}\cot(\sqrt{2}r)$. Take $Y = \xi$, $X = \xi_2$ in (3.3). We have

$$(3.10) \quad \begin{aligned} & 4\beta\xi_3 + h\nabla_{\xi_2}\alpha\xi - hA\phi A\xi_2 - \nabla_{\xi_2}\alpha^2\xi + A^2\phi A\xi_2 \\ & = \beta\{g(\xi_3, S\xi)\xi - \eta(S\xi)\xi_3 + S\xi_3\}. \end{aligned}$$

As α is constant, $S\xi = (4m + h\alpha - \alpha^2)\xi$ and $S\xi_3 = (4m + 6 + h\beta - \beta^2)\xi_3$, from (3.10) we arrive to $\beta\xi_3 = 0$, which is impossible. Thus type (A) real hypersurfaces do not have g -Tanaka-Webster parallel Ricci tensor.

In the case of a type (B) real hypersurface if we take $X = \xi_1$, $Y = \xi$ in (3.3) and bear in mind that $S\xi = (4m + 4 + h\alpha - \alpha^2)\xi$ and $S\phi_1\xi = (4m + 8)\phi_1\xi$, we obtain $\alpha h = 0$, where $\alpha = -2\tan(2r)$. As $\alpha \neq 0$ we must have $h = 0$.

Take then $X = \xi_1$, $Y = \xi_2$ in (3.3) and bear in mind that $h = 0$. With similar computations we obtain $6\beta\xi_3 = 0$, for $\beta = 2\cot(2r)$. As this is impossible, type (B) real hypersurfaces do not have g-Tanaka-Webster parallel Ricci tensor and this finishes the proof of our Theorem.

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